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Linearization of some Poisson–Lie tensor

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Abstract

In this paper we give some example of linear, linearizable and non-linearizable Poisson–Lie structures. We show that any Poisson–Lie tensor P on a Lie group G such that $\mathfrak{g}^*$ is reductive is analytically linearizable, this property being not always satisfied by a general Poisson tensor. We also give examples of linearizable and non-linearizable exact Poisson–Lie structures on some nilpotent Lie groups.

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1. Introduction

Let M be a smooth manifold and P be a Poisson tensor on M: P is a skewsymmetric contravariant 2-tensor field satisfying $[P, P] = 0$ where $[\ ,\ ]$ denotes the Schouten bracket which is the natural extension to skewsymmetric contravariant tensor fields of the bracket of vector fields. Assume that $P(x) = 0$ and denote by $\{x^1, \cdots, x^n\}$ some local coordinates in a neighbourhood of x. Then $\{C_k^{ij} = \partial P^{ij}/\partial x^k|x\}$ defines the structure constants of a Lie algebra $\mathfrak{h}$ called the *linear approximation* [8] of the structure P at the point x.

We may ask whether a given Poisson structure in a neighbourhood of a point where it vanishes is isomorphic to its linear approximation at that point. If so, we shall say that the Poisson structure is *linearizable*.

The first result in this direction is due to V. Arnold who showed that any Poisson structure such that its linear approximation is the non-trivial two-dimensional Lie algebra, is

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linearizable. Weinstein [8] showed that a Poisson structure such that its linear approximation is semisimple, is formally linearizable. Conn [2] proved that if furthermore the Poisson structure is analytic, it is analytically linearizable. Dufour [4] showed that the semisimplicity is not necessary. Molinier [6] showed that any Poisson structure such that its linear approximation is a direct sum of a semisimple ideal and of $\mathbb{R}$, is analytically linearizable.

We are interested in this paper in the linearizability of some particular Poisson tensors vanishing at a point: the Poisson–Lie tensors on a Lie group G. One says that a Poisson tensor P on G is a *Poisson–Lie tensor* if it is multiplicative, i.e. if

$$P(xy) = L_{x*}P(y) + R_{y*}P(x),$$

where L_x (resp. R_x) denotes the left (resp. right) translation by x in G. Such a tensor clearly vanishes at e, the neutral element of G. A group G endowed with such a P is called a *Poisson–Lie group* (G, P).

Let us recall Drinfeld's results [3]: When G is connected and simply connected, there is a bijection between the Poisson–Lie structures on G and the bialgebra structures on the Lie algebra $\mathfrak{g}$ of G. A *bialgebra structure* on a Lie algebra $\mathfrak{g}$ is given by a map $p:\mathfrak{g} \to \Lambda^2\mathfrak{g}$ which is a coboundary (i.e. $p([X, Y]) = \operatorname{ad} X(p(Y)) - \operatorname{ad} Y(p(X))\ \forall X, Y \in \mathfrak{g}$, where ad is the adjoint representation extended to $\Lambda^2\mathfrak{g}$) and which is such that the dual map defines a Lie algebra structure on the dual space $\mathfrak{g}^*$. We denote this bialgebra structure on $\mathfrak{g}$ by $(\mathfrak{g}, p)$ or $(\mathfrak{g}, \mathfrak{g}^*)$. Recall that $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra if and only if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra.

Explicitly, the correspondence runs as follows: if P is a Poisson–Lie tensor on G, $p:\mathfrak{g} \to \Lambda^2\mathfrak{g}$ is defined by the Lie derivative of P at e: $p(X) = (\mathfrak{L}_{\bar{X}}P)(e)$ where $\bar{X}$ is any vector field dual to p, is the linear approximation of P at e.

On the other hand we have:

Lemma 1.1. *The Poisson structure on G is obtained fron the Lie bialgebra structure p on $\mathfrak{g}$ by*

$$P(\exp X) = R_{\exp X*}\frac{e^{\operatorname{ad} X} - 1}{\operatorname{ad} X}p(X) \quad \forall X \in \mathfrak{g}.$$

Hence, a Poisson–Lie structure on a connected and simply connected Lie group G is determined by its linear approximation $\mathfrak{h} = \mathfrak{g}^*$. We study the linearizability of such Poisson–Lie structures (linearizability will be in a neighbourhood of the neutral element of G).

In Section 2, we show that any Poisson–Lie tensor on a connected and simply connected Lie group G such that the linear approximation $\mathfrak{g}^*$ is reductive, is analytically linearizable. We give an example of a non-linearizable Poisson structure on $\mathbb{R}^n$ whose linear approximation is reductive. We also give some more general results on Poisson–Lie structures whose linear approximation are a direct sum with a semisimple factor, and some more general remarks on Poisson structures whose linear approximations are a semi-direct sum with a semisimple factor.

In Section 3, we give examples of linearizable and non-linearizable exact Poisson–Lie structures on some nilpotent Lie groups. Recall that a Poisson–Lie structure P on G is said to be exact if

$$P(x) = L_{x*}Q - R_{x*}Q \quad (x \in G),$$

where Q in a skewsymmetric 2-tensor on the Lie algebra $\mathfrak{g}$. The condition that P is a Poisson tensor is equivalent to the requirement that $[Q, Q] \in \Lambda^3(\mathfrak{g})^{\text{inv}}$.

The results presented here were part of author's Ph.D. Thesis [1].

2. Poisson–Lie tensor in the reductive case

Let G be a Lie group and $\mathfrak{g}$ be the associated Lie algebra.

We are going to study the Poisson–Lie tensors on G which induce on $\mathfrak{g}^*$ a structure of reductive Lie algebra, i.e. $\mathfrak{g}^* = \mathfrak{g}_1 \oplus \Re$ is a direct sum with $\mathfrak{g}_1$ a semisimple ideal and $\Re$ an abelian ideal.

Thus we study the Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$ for such a $\mathfrak{g}^*$, or equivalently, the Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$. We shall denote by q the corresponding cocycle on $\mathfrak{g}^*$; it is defined by $q : \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$ with $\langle q(\eta), X \wedge Y\rangle = \langle \eta, [X, Y]\rangle$. To determine such a q, we observe that its restriction to $\mathfrak{g}_1$ gives a cocycle on a semisimple algebra, hence a 1-coboundary. We also have:

Lemma 2.1. *If $\mathfrak{g}_1$ is a semisimple Lie algebra, then $(\Lambda^2\mathfrak{g}_1)^{\text{inv}\,\mathfrak{g}_1} = 0$.*

From these observations, we easily get:

Proposition 2.2. *Consider a Lie algebra $\mathfrak{g}^*$ such that $\mathfrak{g}^* = \mathfrak{g}_1 \oplus \Re$ is the direct sum with $\mathfrak{g}_1$ a semisimple ideal. Then any 1-cocycle q on $\mathfrak{g}^*$ can be written as*

$$q(X, a) = [(X, a), Q] + \tilde{q}(a) \quad \forall X \in \mathfrak{g}_1 \quad \forall a \in \Re,$$

where $Q \in \Lambda^2(\mathfrak{g}_1 \oplus \Re)$ and $\tilde{q} : \Re \to \Lambda^2(\mathfrak{g}_1 \oplus \Re)$ is a Chevalley 1-cocycle on $\Re$ with values in $\Lambda^2(\mathfrak{g}_1 \oplus \Re)^{\text{inv}\,\mathfrak{g}_1}$.

Denoting by $\tilde{q}^{11}$ the part of $\tilde{q}$ in $\Lambda^2\mathfrak{g}_1$, $\tilde{q}^{12}$ the part of $\tilde{q}$ in $\mathfrak{g}_1 \otimes \Re$ and $\tilde{q}^{22}$ the part of $\tilde{q}$ in $\Lambda^2\Re$, we obtain:

- $\tilde{q}^{11} = 0$,
- $\tilde{q}^{12} = 0$,
- $\tilde{q}^{22} : \Re \to \Lambda^2(\Re)$ *is a 1-cocycle.*

Corollary 2.3. *Consider a Lie bialgebra structure $(\mathfrak{g}, \mathfrak{g}^*)$ such that $\mathfrak{g}^* = \mathfrak{g}_1 \oplus \Re$ is the direct sum with $\mathfrak{g}_1$ a semisimple ideal. Write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{S}$ (direct sum of vector spaces) where $\mathfrak{k} = \{U \in \mathfrak{g} | \langle U, a\rangle = 0\ \forall a \in \Re\}$ and $\mathfrak{S} = \{s \in \mathfrak{g} | \langle s, X\rangle = 0\ \forall X \in \mathfrak{g}_1\}$. Then $\mathfrak{k}$ and $\mathfrak{S}$ are Lie subalgebras of $\mathfrak{g}$.*

Theorem 2.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then any Poisson–Lie tensor on G which induces on $\mathfrak{g}^*$ a structure of reductive Lie algebra, i.e. $\mathfrak{g}^* = \mathfrak{g}_1 \oplus \Re$ is the direct sum where $\mathfrak{g}_1$ a semisimple ideal and $\Re$ an abelian ideal, is analytically linearizable.*

Proof. We use the same notations as in Corollary 2.3. Writting that p^* is a Lie bracket on $\mathfrak{g}^*$, $p = {}^t[\ ,\ ]_{\mathfrak{g}_1 \oplus \mathfrak{R}}$ we get

$$\langle p(U,s), (X,a) \wedge (Y,b) \rangle = \langle U, [X,Y] \rangle$$
$$U \in \mathfrak{k},\quad s \in \mathfrak{G},\quad X, Y \in \mathfrak{g}_1,\quad a, b \in \mathfrak{R}.$$

So $p(\mathfrak{k}) \subset \Lambda^2 \mathfrak{k}$ and $p(\mathfrak{G}) = \{0\}$.

Since we consider a Poisson–Lie tensor P, it is multiplicative, so $P(xy) = L_{x*}P(y) + R_{y*}P(x)$.

Locally in a neighbourhood of e in G, any $g \in G$ can be written $g = \exp U \exp s$ with $U \in \mathfrak{k}$ and $s \in \mathfrak{G}$. Thus

$$P(g) = P(\exp U \exp s) = R_{\exp s_*} R_{\exp U_*} \frac{\mathrm{e}^{\operatorname{ad} U} - 1}{\operatorname{ad} U} p(U).$$

If $\{U_1, \cdots, U_n\}$ is a base of $\mathfrak{k}$ and if $\{s_1, \cdots, s_r\}$ is a base of $\mathfrak{G}$, the local coordinates on G will be given by $(x^1, \cdots, x^n, x^{n+1}, \cdots, x^{n+r})$ if $g = \exp(\sum_{1 \le i \le n} x^i U_i) \exp(\sum_{1 \le i \le r} x^{n+i} s_i)$. Since $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$ (see Corollary 2.3), we have in these local coordinates:

$$P(g) = P(\exp U \exp s) = \sum_{1 \le i,j \le n} P^{ij} \frac{\partial}{\partial x^i}\Big|_{\exp U \exp s} \wedge \frac{\partial}{\partial x^j}\Big|_{\exp U \exp s}$$

and P^{ij} depends on x^k for $k = 1, \cdots, n$.

We define $P' = \sum_{1 \le i,j \le n} P^{ij} \partial/\partial x^i \wedge \partial/\partial x^i$ a Poisson tensor on $\mathbb{R}^n$. We have $P'(0) = 0$ and $C_k^{ij} = \partial P^{ij}/\partial x^k|0$ are the structure constants of the Lie algebra $\mathfrak{g}_1$ which is semisimple. From the theorem of linearization of Conn [2] in the analytic case, P' is linearizable by an analytic change in the variables $(x^1, \cdots, x^n)$. Hence, by the same change in $(x^1, \cdots, x^n)$ and the identity on $(x^{n+1}, \cdots, x^{n+r})$, P is linearized.

Remark 2.5. We are going to see that the result does not always hold for general Poisson tensors (hence, the stress on the Poisson–Lie condition in Theorem 2.4).

Let P be a Poisson structure on a manifold M, which vanishes at a point $y \in M$ and such that $\mathfrak{h}$, its linear approximation at x, is reductive, i.e. $\mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{R}$ with $\mathfrak{g}_1$ semisimple and $\mathfrak{R}$ abelian, with $\dim \mathfrak{R} \ge 2$. The case where $\dim \mathfrak{R} = 1$, i.e. $\mathfrak{R} = \mathbb{R}$, is known [6] and in that case P is analytically linearizable.

We consider P, a Poisson tensor on $\mathbb{R}^{n+r}$ (where $n = \dim \mathfrak{g}_1$ and $r = \dim \mathfrak{R}$, $r \ge 2$), vanishing at 0 and such that the linear structure is $\mathfrak{g}_1 \oplus \mathfrak{R}$, given by

$$P^{ij}(x) = \sum_{1 \le k \le n} C_k^{ij} x^k + B^{ij},$$

where the C_k^{ij} are the structure constants of $\mathfrak{h} = \mathfrak{g}_1 \oplus \mathfrak{R}$ in the basis $\rangle e_1, \cdots, e_{n+r}\langle$ where $e_i \in \mathfrak{g}_1\ \forall i \le n$ and $e_j \in \mathfrak{R}\ \forall j > n$; and where B^{ij} are the non-linear terms of P^{ij}, and we assume that $B^{ij} = 0$ for i or $j \le n$.

Then if B is not zero, the maximal dimension of the symplectic leaves for the linear structure is different from the maximal dimension of the symplectic leaves for P, so P is not linearizable.

Theorem 2.6. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Any Poisson–Lie tensor P on G such that $\mathfrak{g}^*$, the dual of $\mathfrak{g}$, is written $\mathfrak{g}^* = \mathfrak{g}_1 \oplus \mathfrak{R}$ the direct sum where $\mathfrak{g}_1$ is a semisimple ideal of dimension n and $\mathfrak{R}$ is an ideal of dimension r is linearizable if and only if the induced Poisson structure on the group corresponding to $\mathfrak{G} = \{s \in \mathfrak{g} | \langle X, s\rangle = 0\ \forall X \in \mathfrak{g}_1\}$ (which is a Lie subalgebra) is linearizable.*

Furthermore, one can seperate the variables, i.e. find local coordinates $(x^1, \cdots, x^{n+r})$ and write $P = L \oplus T$ where L is a linear Poisson tensor corresponding to the structure constants of the Lie algebra $\mathfrak{g}_1$, $L = \sum_{1\le i,j,k\le n} C_k^{ij} x^k \partial/\partial x^i \wedge \partial/\partial x^j$, and where

$$T = \sum_{n+1\le i,j\le n+r} T^{ij}(x^{n+1}, \cdots, x^{n+r}) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j}.$$

The proof is analogous to the proof of Theorem 2.4.

Remark 2.7. This result does not always hold for general Poisson structures.

Using the fact that $H^2(\mathfrak{g}_1 \bowtie \mathfrak{R}, \rho, V) = H^2_{\mathfrak{g}_1-\text{equivariant}}(\mathfrak{R}, \rho_2, V)$ which is a particular case of the theorem of Hochschild and Serre [5] (where $\mathfrak{h} = \mathfrak{g}_1 \bowtie \mathfrak{R}$ denotes the semi-direct sum of a semisimple Lie algebra $\mathfrak{g}_1$ and of an ideal $\mathfrak{R}$), we get:

Let Λ be a Poisson structure on a manifold M, vanishing at a point x and such that the associated linear structure at x is written $\mathfrak{h} = \mathfrak{g}_1 \bowtie \mathfrak{R}$, the semi-direct sum of a semisimple Lie algebra of dimension n and of an ideal of dimension r.

Then there exist formal coordinates $(x^1, \cdots, x^n, x^{n+1}, \cdots, x^{n+r})$ in a neighbourhood of x such that:

$$\{x^i, x^j\} = \sum_{1\le k\le n} C_k^{ij} x^k \quad \text{for } i \text{ and } j \le n,$$

$$\{x^i, x^j\} = \sum_{n+1\le k\le n+r} C_k^{ij} x^k \quad \text{for } i \in \{1, \cdots, n\} \text{ and } j \in \{n+1, \cdots, n+r\},$$

$$\{x^i, x^j\} = \sum_{n+1\le k\le n+r} C_k^{ij} x^k + B^{ij}(x^1, \cdots, x^n, x^{n+1}, \cdots, x^{n+r}) \quad \text{for } i \text{ and } j \in \{n+1, \cdots, n+r\},$$

where the C_k^{ij} are the structure constants of the Lie algebra $\mathfrak{g}_1 \bowtie \mathfrak{R}$.

This result can be found in [7].

3. Exact Poisson–Lie structures in the nilpotent case

In this section we study the linearizability of some Poisson–Lie structures on a given Lie group G. The stress, here is on G and not on $\mathfrak{g}^*$.

Consider an exact Poisson–Lie structure on a Lie group G, it is given by $P_x = L_{x*}Q - R_{x*}Q$ ($x \in G$), where Q is skewsymmetric 2-tensor on the Lie algebra $\mathfrak{g}$ such that $[Q, Q] \in \Lambda^3(\mathfrak{g})^{\text{inv}}$.

Lemma 3.1. *Denote by $\{X_1, \cdots, X_n\}$ a basis of $\mathfrak{g}$ and by $\{x^1, \cdots, x^n\}$ the local coordonates in the logarithmic chart. An exact Poisson–Lie tensor P on the Lie group G is a formal series composed with monomials in $\{x^1, \cdots, x^n\}$ of odd degree and is given by:*

$$P(\exp X) = \sum_{1\le k,l\le n} Q^{kl} \sum_{1\le i\le n} \sum_{1\le j\le n} \left\{ (\delta_i^k (\text{ad}\, X\, X_l)^j) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \frac{1}{12} (\text{ad}\, X\, X^k)^i (\text{ad}^2\, X\, X_l)^j \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} + \cdots \right\}$$

Assume now that G is nilpotent, then P is polynomial.

Recall the following definition:

Definition 3.1. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Define $\mathfrak{g}^{i+1} = [\mathfrak{g}^i, \mathfrak{g}]$ with the convention that $\mathfrak{g}^0 = \mathfrak{g}$. Then $\mathfrak{g}$ is a *m-step nilpotent* algebra if $\mathfrak{g}^m = 0$.

One can easily see that:

Proposition 3.2. *Any exact Poisson–Lie tensor on a two-step nilpotent Lie group is linear.*

Proposition 3.3. *Any exact Poisson–Lie tensor on a three-step nilpotent Lie group of dimension less or equal to 6 is linearizable.*

Proof. P is composed of its linear part and polynomials in $\{x^1, \cdots, x^6\}$ of degree 3. The idea is to eliminate the polynomials of degree 3 by a change of coordinates of the form: $y^i = x^i + f^i(x)$ where f^i is a homogeneous polynomial of degree 3. We determine the f^i solutions of

$$P_{(3)}^{ij}(x) + \sum_{1\le k,l\le n} \left(\frac{\partial P_1^{ik}}{\partial x^l} x^l \frac{\partial f^j}{\partial x^k} - \frac{\partial P_{(1)}^{ij}}{\partial x^l} x^l \frac{\partial f^i}{\partial x^k} \right) - \sum_{1\le k\le n} \frac{\partial P_{(1)}^{ij}}{\partial x^k} f^k(x) = 0 \quad \forall i, j \tag{1}$$

For each of the three-step nilpotent Lie group of dimension less or equal to 6, we find solutions of these equations. □

Remark 3.4. We show now that there exist exact Poisson–Lie tensors on a four-step nilpotent Lie group (of dimension 5) which are not linearizable.

Consider the algebra $\mathfrak{g} = L_4 = \rangle X_1, X_2, X_3, X_4, X_5 \langle$ where the non-zero brackets are given by:

$$[X_1, X_2] = X_3; \qquad [X_1, X_3] = X_4; \qquad [X_1, X_4] = X_5.$$

This is a four-step nilpotent Lie algebra of dimension 5.
First, we study the condition $[Q, Q]$ ad-invariant:

$$[X, [Q, Q]] = 0 \ \forall X \in \mathfrak{g} \quad \Longleftrightarrow \quad \begin{cases} Q^{12} = 0, \\ Q^{13} = 0, \\ Q^{14} = 0, \\ Q^{15} Q^{23} = 0. \end{cases}$$

Then consider the particular case where:
(1) $Q = Q^{23} X_2 \wedge X_3$, i.e. $Q^{ij} = 0$ except Q^{23}. Then $[Q, Q] = 0$ but P is not linearizable. (Eq. (1), in this case, has no solution).
(2) $Q^{23} = 0$. Then P is linearizable. (A solution of Eq. (1) is given by $f^1 = f^2 = f^3 = f^4 = 0$ and $f^5 = -\frac{1}{12}(x^1)^2 x^3$).

Hence, we get:

Proposition 3.5. *On the connected and simply connected Lie group of Lie algebra L_4, there exist some exact Poisson–Lie structures (G, P) where P is given by $P(x) = L_{x*}Q - R_{x*}Q$ such that*
- $[Q, Q] = 0$ *and P is linearizable,*
- $[Q, Q] = 0$ *and P is non linearizable,*
- $[Q, Q]$ *is ad-invariant and P is linearizable.*

References

[1] V. Chloup-Arnould, Groupes de Lie–Poisson, Thése de l'universitè de Metz (1996).
[2] J.-F. Conn, Normal forms for analytic Poisson structures, Ann. of Math. 119 (1984) 577–601.
[3] V.G. Drinfeld, Hamiltonian structure on Lie groups and the geometric meaning of the classical Yang–Baxter equation, Soc. Math. Dokl. 27 (1) (1983) 68–71.
[4] J.P. Dufour, Linéarisation de certaines structures de Poisson, J. Differential Geom. 32 (5) (1990) 415–428.
[5] G. Hochschild and J.-P. Serre, Cohomology of Lie algebras, Ann. of Math. 57 (3) (1953) 591–603.
[6] Molinier, Linéarisation de structures de Poisson, Thèse de l'université de Montpellier II (1993).
[7] A. Wade, Normalisation de structures de Poisson, Thèse de l'université de Montpellier II (1996).
[8] A. Weinstein, The local structure of Poisson manifolds, J. Differential Geom. 18 (1983) 523–557.